Burke Situation 2: Fraction Operations and Models – It's the Law!

~~Prompt~~:

In a class for secondary teachers, a fictional 8th grader named Pesky was introduced by the teacher into an online chat room. The class had been studying number structures and Pesky posed a basic question: "Look, you've been trying to teach me fractions ever since I came to this school. If you would just cut the word problems and stick to the math, with a few minor changes, I could do this. You gotta change the way you add fractions. If you would just let me add tops and add bottoms I would never forget how to add fractions. You wouldn't have to keep teaching us the same stuff every year. We'd get it the first time!" The students participating in the chat room were asked to help hapless Pesky understand why adding fractions the school's way was the better way.

Many offered explanations like the following one.

> "Ah Pesky, your method doesn't make any sense. Suppose you have a pizza and you take a fourth of it while your sister and her friends take two thirds of it. How much of the pizza will be gone? Doing this your way, you would say 3/7 since 1/4+2/3 = 3/7. But 3/7 is less than half of the pizza. Hey, your sister and her friends alone took more than half of the pizza! Now if the pizza was sliced into 12 equal pieces, which is a common multiple of 4 and 3, then 1/4 of the pizza would be equal to 3 pieces or 3/12 of the pizza. And 2/3 of the pizza would be equal to 8 pieces or 8/12 of the pizza. So together, 3/12 + 8/12 is 11 pieces which means that 11/12 of the pizza is gone. So, the school method is better."

Pesky (i.e. the teacher) responded:

> "You mean, 'So the school method works in that situation.' That doesn't make it better. I can give you situations where my method works and yours doesn't. Suppose my sister had three pizzas and she and her friends ate two of them. Suppose I had four pizzas and I ate 1 of them. Then all together we ate a total of three pizzas out of 7 pizzas. Not 11 out of 12 pizzas! So my method works for some word problems and yours works for others. But we should drop the word problems and stick to the math. My method makes more sense mathematically and it is way, way easier. "

The student responded:

> "You forget one thing about the school method, Pesky. It's the law!"

Commentary:

Very often textbooks rationalize fraction operations by invoking area models, as in the case of the pizza, or other real world models. Pesky supports his algorithm for adding fractions ($\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$) using real world models as well. He makes a valid point about the ease of remembering his way of adding fractions. The challenge for the teacher is to understand mathematically why we add fractions and rational numbers the way we do.

Many mathematical foci emerge from this prompt. First of all, mathematically speaking, the addition algorithm for fractions is not unique in the realm of school mathematics for being notoriously difficult for students to remember. There are many other such “laws” ratified in textbooks and standardized tests that our children are expected to master and remember. They include division algorithms, multiplication algorithms, and laws of negative numbers, to name a few. Each of these algorithms deserves deep understanding on the part of the teacher as well as a few alternative self-created algorithms like Pesky’s. “Yours is not to reason why, so just ‘Invert and Multiply’.”

A second mathematical foci concerns the historical representations and meanings of numbers. Indeed, historically, negative numbers, irrational numbers and rational fractions were not always thought of as numbers. Until Diophantus the Greeks treated fractions as ratios of two numbers. Indeed, in the above prompt, Pesky is treating rational fractions not as single numbers but as ratios of two whole numbers and is proposing a method for adding ratios. Under the umbrella of the geometric metaphor we know as the real number line, these various notions of numbers became accepted, unified and systematized. Thus, the meaning of our numbers is interwoven with the meaning of our operations on these numbers and both have deep historical roots.

A third mathematical focus is the notion that fractions can be defined as numbers in various ways relative to other system of numbers and these definitions determine how the operations we prescribe on fractions will behave. For example, if real numbers and their axioms are taken as primitives and if integers and rational numbers are defined within this system as special subsets of real numbers, then the rules for operating on these special subsets of real numbers using real number addition and multiplication (and subtraction and division) can be proven from the axioms to be properties of these operations when restricted to these subsets of numbers.

A fourth focus is found in the wide variety of interpretations of rational numbers. Students are exposed to many such models in their work with fractions over the years. However, teachers need to understand how these models can reinforce student conceptions and understanding of rational number. Teachers using the area model, for example, to help students make sense of the rule for multiplying fractions, often encounter students who say: "Why do I need to know this? I already know how to multiply fractions."

Mathematical Foci:

(These commentary following each foci are not complete but suggest a direction for discussion.)

Mathematical Focus 1.

There are many algorithms for operations that are learned as though they are mathematical laws. Yet alternative algorithms for binary operations can be invented by students and traditional algorithms can be justified in sensible ways.

I was thinking here of development of operation and algorithms that we emphasize in our mathematics for elementary teachers courses but never expose our secondary preservice teachers to. There is a need to address these same issues at a little higher level, including real and complex numbers and operations, in the curriculum for preservice secondary teachers.

Mathematical Focus 2.

The meaning of our numbers is interwoven with the meaning of our operations on these numbers and both have deep historical roots.

This focus can describe the historical roots of fraction operations. If anything, this history shows a pattern of complexity and avoidance with great irregularities in notations. The historical emergence of decimal fractions is important as well. The development of a child's understanding of the rational number system could perhaps be one of those areas where "Ontogeny recapitulates Phylogeny" is an apt analogy.

Mathematical Focus 3.

Fractions can be defined as numbers in various ways relative to other system of numbers and these definitions determine how the operations we prescribe on fractions will behave.

Starting with the real numbers taken as given and understood, along with the axioms of the real number system as an ordered field, the natural numbers and the integers can be defined within the real numbers. Then the rational number *a/b* where *a* and *b* are integers and $b \neq 0$ can be defined as the real number ab^{-1} or *a* times the multiplicative inverse of *b*. From this point on the axioms of the system determine the rules for addition and multiplication when they are restricted to the set of rational numbers.

A more classic development starts with integers and defines the rational numbers as the "quotient field" of the integers. Intuitively, this corresponds to extending the number system of integers by including exactly those numbers necessary so that the division operation is closed on the set of all non-zero numbers in the system. In this construction or definition of the rational numbers, since the goal is to define the operations of addition and multiplication of rationals in a way that yields integer addition and multiplication when restricted to the set of integers, the rules for operating with rational numbers are determined. Specifically, Pesky's form of addition is ruled out.

Mathematical Focus 4.

Rational fractions (fractions of the form a/b where a and b are integers with b≠0) have a wide variety of interpretations. These interpretations lend themselves to various interpretations of the operations on fractions. The interpretations of the operations in turn can help students make sense out of the algorithms for those operations.

There is a pragmatic side to the evolution of mathematics. While some marvel at the "unexpected" applicability of mathematics, others disagree and note that mathematics is founded on abstract models of human logical processes that have proven to be useful to humans in their interactions with the world. What this means for our children is that, to some extent, the numbers and operations and algorithms that they are taught are naturally selected because they are useful and not because they describe some immutable truth. In the case of Pesky, his definition of addition of fractions, while a well-defined binary operation, fails as a choice for defining fraction addition because it is not as useful as the standard definition. Indeed it can be argued that historically it is because of the word problems that embedded the applications humans valued most that our number systems evolved the way they did. Pesky knows that by dropping all the word problems and hence applications of rational numbers from consideration, a strong support for the traditional definition of addition of fractions and its study in our schools is lost.